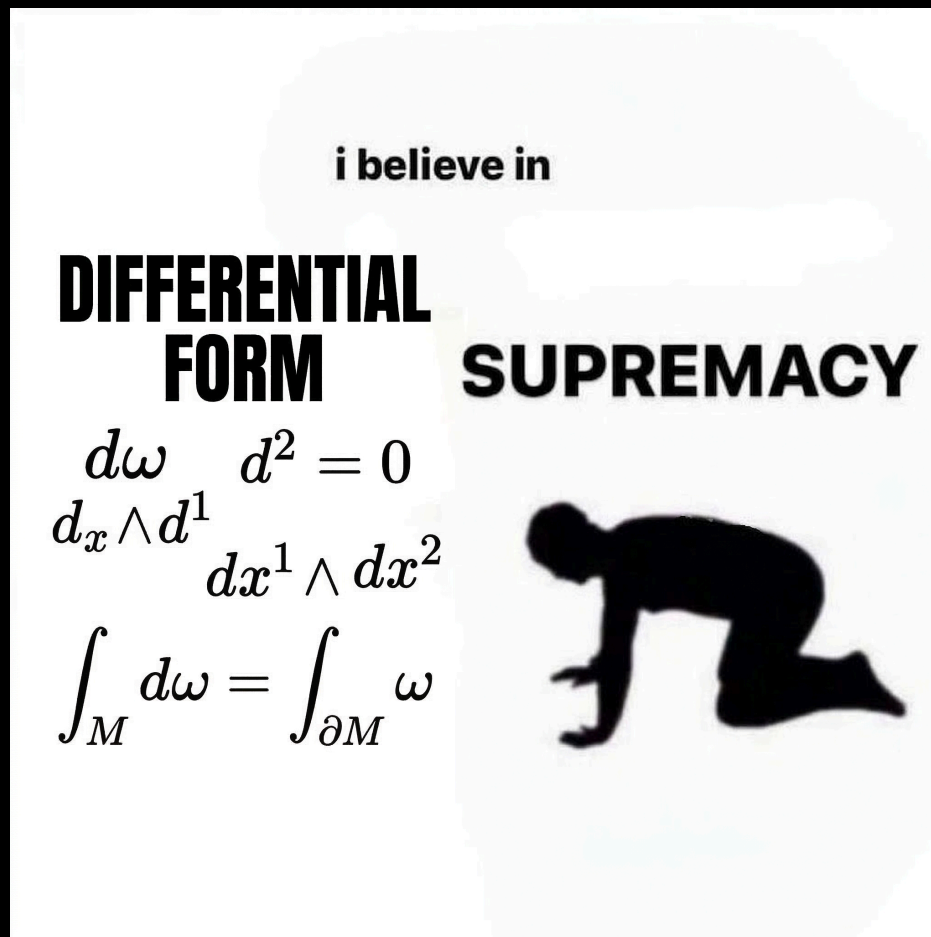


Ψ ec: A local spectral exterior calculus 2021

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Exterior Calculus + Fourier Transform



Vector Calculus is everywhere

Maxwell's Equations

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\operatorname{div} \mathbf{B} = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Linear Elasticity

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^\top)$$

Navier-Stokes (incompressible)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \operatorname{grad}) \mathbf{u} = -\frac{1}{\rho} \operatorname{grad} p + \nu \Delta \mathbf{u} + \mathbf{f}$$

$$\operatorname{div} \mathbf{u} = 0$$

Structural Identities

The nature of PDEs

$$\mathbf{curl} \circ \mathbf{grad} = 0 \qquad \mathbf{div} \circ \mathbf{curl} = 0$$

- Violate Structural Identities $\rightarrow$ Numerical Scheme breaks!
 - Spurious Solutions
 - Unphysical Fields
 - Magnetic Monopoles

The de Rham Complex

Connecting the Differentials

$$H(\mathbf{grad}; \Omega) \xrightarrow{\mathbf{grad}} H(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} H(\mathbf{div}; \Omega) \xrightarrow{\mathbf{div}} L^2(\Omega)$$

$$\mathbf{curl} \circ \mathbf{grad} = 0 \qquad \mathbf{div} \circ \mathbf{curl} = 0$$

The Discretization Frameworks

Structure Preservation is key!

FEEC / DEC

- Mesh-based
- Continuous forms are replaced by **discrete surrogates** (cochains)
- Structure preservation must be **carefully constructed**

Ψ_{ec}

- Mesh-free
- Basis functions are **genuine continuous forms**
- Structure preservation is **automatic**.

The de Rham Complex

Everything is a Differential Form

$$H(\mathbf{grad}; \Omega) \xrightarrow{\mathbf{grad}} H(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} H(\mathbf{div}; \Omega) \xrightarrow{\mathbf{div}} L^2(\Omega)$$

$$\mathbf{curl} \circ \mathbf{grad} = 0 \quad \mathbf{div} \circ \mathbf{curl} = 0$$

$$H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} H\Lambda^2 \xrightarrow{d} H\Lambda^3(\Omega)$$

$$d^2 = d \circ d = 0$$

What is a Differential Form?

Differential k -form $\omega \in \Lambda^k(\Omega)$ is a k -dimensional **integrand**!

Encodes **orientation** and **antisymmetry**.

$$\text{point/scalar : } \alpha = x^3 - 5x^2 + 4 \in \Lambda^0(\Omega)$$

$$\text{curve/circulation : } \beta = 3 \sin(x) dx + e^{xy} dy + \log(z) dz \in \Lambda^1(\Omega)$$

$$\text{surface/flux : } \gamma = 5 dx \wedge dy + 2 dx \wedge dz + 2 dy \wedge dz \in \Lambda^2(\Omega)$$

$$\text{volume/density : } \delta = \cos(xy) dx \wedge dy \wedge dz \in \Lambda^3(\Omega)$$

Unifies integration over k -dimensional submanifold $M \subseteq \Omega$.

$$\int_M \omega \quad \dim(M) = k \quad \omega \in \Lambda^k(\Omega)$$

The Exterior Derivative

One derivative to rule them all.

$$d_k : H\Lambda^k \rightarrow H\Lambda^{k+1}$$

$$d^2 = d \circ d = 0$$

$$\text{grad } f = (d_0 f)^\#$$

$$\text{curl } u = (\star d_1 u^b)^\#$$

$$\text{div } u = \star d_2 \star u^b$$

Differential and Codifferential

Codifferential

$$\delta : \Lambda^k(\Omega) \rightarrow \Lambda^{k-1}(\Omega)$$

Defined as L^2 -adjoint

$$\delta := d^*$$

$$\langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, \delta\beta \rangle_{L^2}$$

Together build the Hodge-Laplacian

$$\Delta = d\delta + \delta d$$

d	δ
grad	$-\text{div}$
curl	curl
div	$-\text{grad}$

Hodge-Helmholtz Decomposition

on trivial topologies

Every k -form splits as

$$\omega = \underbrace{d\alpha}_{\text{exact}} + \underbrace{\delta\beta}_{\text{coexact}}$$

Exact

$$\exists \beta : d\beta = \alpha$$

image of d

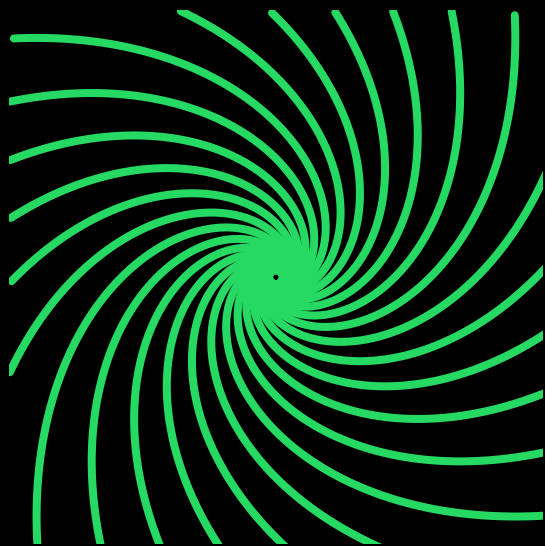
Coexact

$$\exists \beta : \delta\beta = \alpha$$

image of δ

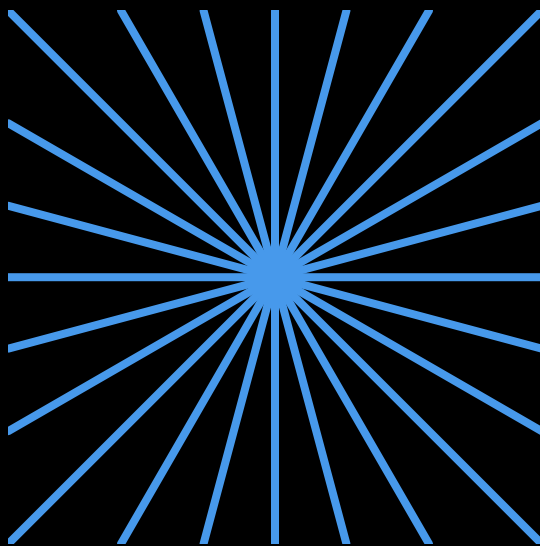
Helmholtz Decomposition of Vector Fields

Mixed



$$[x - y, y + x]$$

curl-free (exact)



$$[x, y]$$

div-free (coexact)



$$[-y, x]$$

The Fourier Transform

Why relevant for PDEs?

Turns differentiation into multiplication.

$$\frac{d}{dx} \mapsto i\xi$$

Differential equations become algebraic equations!

$$-u'' + \lambda u = f \quad \mapsto \quad (\xi^2 + \lambda)\hat{u} = \hat{f}$$

Does this also happen in higher dimensions?

Vector Calculus becomes Vector Algebra

in the Frequency Domain

$$\mathbf{grad} f = \nabla f \quad \mapsto \quad i\xi \hat{f}$$

$$\mathbf{curl} u = \nabla \times u \quad \mapsto \quad i\xi \times \hat{u}$$

$$\mathbf{div} u = \nabla \cdot u \quad \mapsto \quad i\xi \cdot \hat{u}$$

Exterior Calculus becomes Exterior Algebra

in the Frequency Domain

$$d\omega \mapsto \iota_{i\xi} \hat{\omega}$$

$$\delta\omega \mapsto i\xi \wedge \hat{\omega}$$

where

- $\iota_{i\xi}$ is the **interior product** (contraction) with $i\xi$
- $i\xi \wedge \cdot$ is the **wedge product** (extension) with $i\xi$

The Spherical Geometry of Derivatives

in the Frequency Domain

Write the frequency vector in spherical coordinates:

$$\xi = |\xi| d\hat{r}$$

where $\hat{r}$ is the unit radial frequency basis. Then:

$$\hat{d}\hat{\omega} = i|\xi| \iota_{\partial/\partial\hat{r}} \hat{\omega}$$

$$\hat{\delta}\hat{\omega} = i|\xi| \partial/\partial\hat{r} \wedge \hat{\omega}$$

Both operators factor into a scalar part $i|\xi|$ and a purely geometric part that acts only in the **radial direction**.

To be, or not to be radial

- The contraction $\iota_{\partial/\partial\hat{r}}$ strips off the radial component $\partial/\partial\hat{r}$ from a form.

$$\hat{\omega}_{r\theta} \partial/\partial\hat{r} \wedge \partial/\partial\hat{\theta} \xrightarrow{\hat{d}} \hat{\omega}_{r\theta} \partial/\partial\hat{\theta}$$

- The wedge $\partial/\partial\hat{r} \wedge$ attaches a radial component.

$$\hat{\omega}_{\theta} \partial/\partial\hat{\theta} \xrightarrow{\hat{\delta}} \hat{\omega}_{\theta} \partial/\partial\hat{r} \wedge \partial/\partial\hat{\theta}$$

Magic

All of multidimensional calculus just becomes adding and removing radial components!

Trivial Structural Identities

You cannot remove a radial component twice.

$$\hat{d} \circ \hat{d} = \iota_{\partial/\partial \hat{r}} \iota_{\partial/\partial \hat{r}} = 0$$

You cannot have two radial component by antisymmetry.

$$\hat{\delta} \circ \hat{\delta} = \partial/\partial \hat{r} \wedge \partial/\partial \hat{r} = 0$$

Laplacian: Geometric parts compose to identity, leaving only a scalar.

$$\hat{\Delta} = \hat{d}\hat{\delta} + \hat{\delta}\hat{d} = |\xi|^2$$

Trivial Hodge-Helmholtz Decomposition

via Fourier

Hodge-Helmholtz becomes basic geometric fact.

All forms split into two orthogonal types:

- **Tangential/Exact forms** $\hat{\omega}_d$
- **Radial/Coexact forms** $\hat{\omega}_\delta$

The differentials move between them:

$$\hat{d} : \text{radial} \xrightarrow{\iota_{\partial/\partial \hat{r}}} \text{tangential}$$

$$\hat{\delta} : \text{tangential} \xrightarrow{\hat{r} \wedge} \text{radial}$$

Spectral exterior calculus to solve PDEs

Numerical methods? $\rightarrow$ We need to construct basis functions!

Requirements:

- meshless
- continuous basis functions $b \in C^\infty$

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A first idea

Inspired by Fourier: Use plane waves in space.

$$b(x) = e^{i\xi_0 x} = \cos(\xi_0 x) + i \sin(\xi_0 x)$$

→ classical spectral method.

The Localization Tradeoff

Position vs Frequency

Plane wave — perfect frequency

$$e^{i\xi_0 x} \quad \Leftrightarrow \quad \delta(\xi - \xi_0)$$

✗ Zero position localization.

Classical spectral method.

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Dirac delta — perfect position

$$\delta(x - x_0) \quad \Leftrightarrow \quad e^{ix_0 \xi}$$

✗ Zero frequency localization.

Spirit of finite elements.

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Heisenberg Uncertainty Principle

Any function occupies a region of area $\geq \frac{1}{2}$ in the position $\times$ frequency space.

$$\Delta x \Delta \xi \geq \frac{1}{2}$$

Strike a Balance

Need localization in **both**.

→ Handle global/periodic and local/singular phenomenon alike.

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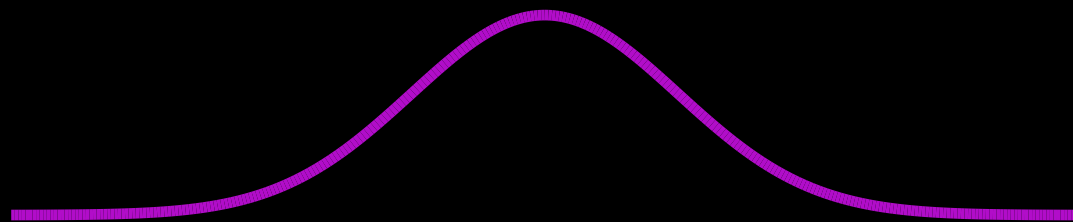
The Gaussian

- Same localization in both domains, minimal uncertainty overall.

$$\Delta x \Delta \xi = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2}$$

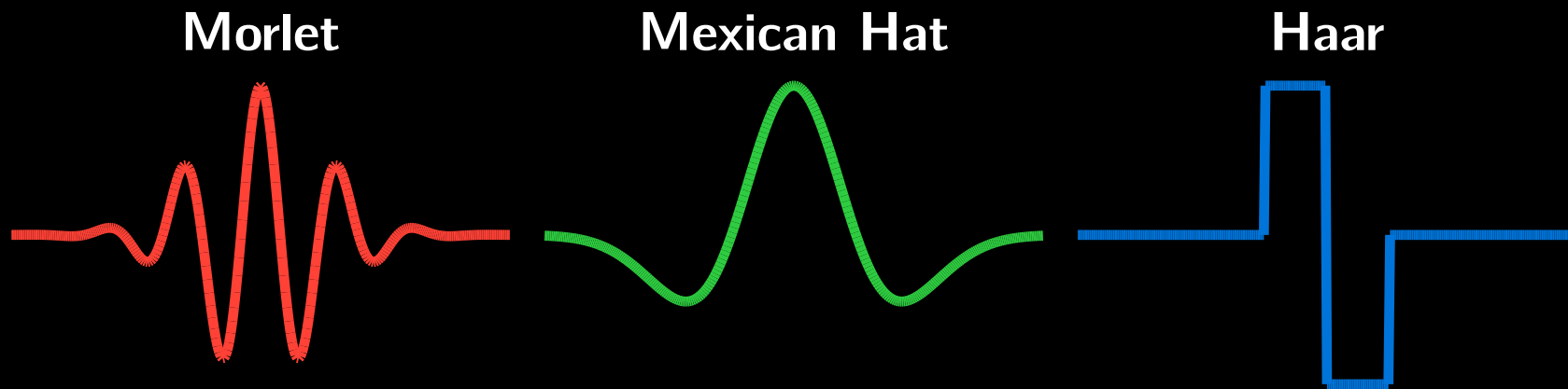
- Fourier self-dual.

$$e^{-x^2/2} \xleftrightarrow{\text{Fourier}} e^{-\xi^2/2}$$



Wavelets

Functions localized in both space and frequency



Wavelets can be scaled and translated.

- large scale + narrow in frequency → smooth/periodic phenomena
- small scale + wide in frequency → sharp/singular phenomena

Polar Wavelets

Exterior Calculus is spherical in frequency space.

→ Define wavelets in spherical frequency coordinates $\xi = (\rho, \theta)$

$$\hat{\psi}(\rho, \theta) = h(\rho) \cdot g(\theta)$$

- $h(\rho)$ radial factor: where d and δ act
- $g(\theta)$ angular factor: directionality; from isotropic to curvelet

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The Hodge decomposition is **built-in**:

Tangential/Exact Wavelets

$$\hat{\psi}^d \in \text{im}(d)$$

Radial/Coexact Wavelets

$$\hat{\psi}^\delta \in \text{im}(\delta)$$

Differential Form Wavelets

$$\psi_s^{k,\nu}$$

- form degree k
- type $\nu \in \{d, \delta\}$
- geometry $s = (j, k, t)$ — scale, translation, orientation.

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Wavelets are built with de Rham complex in mind!

→ Differentials map one wavelet to another. Geometry s unchanged!

→ Structural Identities elegantly fulfilled.

$$d\psi_s^{k,\delta} = \psi_s^{k+1,d} \qquad d\psi_s^{k,d} = 0$$

The Main Message

multidimensional calculus



exterior calculus



frequency space



spherical coordinates

The de Rham complex reduces to:

Radial vs Tangential

Polar wavelets respect this by construction.

→ **Structure preservation** is free.

The End.

Frequency Space $\hat{\mathbb{R}}^n$

The dual space of position space

The Fourier exponential $e^{ix^p \xi_p}$ must be coordinate-independent.

Under $x^p \mapsto \lambda x^p$ we need $\xi_p \mapsto \xi_p / \lambda$:

x^p **contravariant**

position, vector,
transforms **with** basis change

ξ_p **covariant**

frequency, covector,
transforms **against** basis change

Dual space

$\hat{\mathbb{R}}^n = (\mathbb{R}^n)^*$. The pairing $\langle \xi, x \rangle = \xi_p x^p$ is canonical $\rightarrow$ **no metric required**.

ξ is a **stack of parallel hyperplanes** (wavefronts of $e^{i\xi_p x^p}$). The pairing counts how many wavefronts x crosses.

The Fourier Transform Changes Variance

Covariant spatial forms become contravariant frequency multivectors

Spatial forms

Built from $dx^i \in T^*\mathbb{R}^n$

Covariant — eat tangent vectors

Frequency forms

Built from $\partial/\partial\xi_i \in T\hat{\mathbb{R}}^n$

Contravariant — eat covectors

The Fourier transform is a map between a vector space and its dual. In flat $\mathbb{R}^n$ these are identified by the metric, but the distinction is real.

The Fourier Transform Changes Degree

Integration in space becomes point evaluation in frequency

An n -form integrated over $\mathbb{R}^n \rightarrow$ a 0-form evaluated at a point.

$$\int_{\mathbb{R}^n} f(x) \, dx^1 \wedge \cdots \wedge dx^n = (2\pi)^{n/2} \hat{f}(0)$$

Degree flip

$$\mathcal{F} : \Lambda^k(\mathbb{R}^n) \rightarrow \hat{\Lambda}^{n-k}(\hat{\mathbb{R}}^n)$$

The k occupied directions are consumed by integration. The remaining $n - k$ become the basis of the output form in frequency.